ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Computational Geometry 33 (2006) 99–105

Computational Geometry
Theory and Applications

www.elsevier.com/locate/comgeo

Progress on maximum weight triangulation ☆

Jianbo Qian [a,*], Cao An Wang [b]

[a] *School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*
[b] *Department of Computer Science, Memorial University of Newfoundland, St. John's, Newfoundland, Canada A1B 3X5*

Received 24 October 2003; received in revised form 8 July 2005; accepted 11 November 2005

Available online 11 January 2006

Communicated by S. Akl

Abstract

In this paper, we investigate various properties and problems associated with the maximum weight triangulation of a point set in the plane. We prove that the weight of a maximum weight triangulation of any planar point set with diameter D is bounded above by $((2\varepsilon+2)\cdot n+\frac{\pi(1-2\varepsilon)}{8\varepsilon\sqrt{1-\varepsilon^2}}+\frac{\pi}{2}-5(\varepsilon+1))D$, where ε is any constant $0<\varepsilon\leqslant\frac{1}{2}$ and n is the number of points in the set. If we use the so-called spoke-scan algorithm to find a triangulation of the point set, we obtain an approximation ratio of 4.238. Furthermore, if the point set forms a semi-circled convex polygon, then its maximum weight triangulation can be found in $O(n^2)$ time.

Keywords: Algorithm; Approximation; Maximum weight triangulation

1. Introduction

Triangulation of a set of points is a fundamental structure in computational geometry. Among different triangulations, the *minimum weight triangulation* (*MWT* for short) of a set of points in the plane attracts special attention [3,6–8]. The construction of the *MWT* of a point set is still an outstanding open problem. When the given point set is the set of vertices of a convex polygon (so-called *convex point set*), then the corresponding *MWT* can be found in $O(n^3)$ time by dynamic programming [3,6]. There are many approximation algorithms for *MWT*. A constant ratio approximation algorithm of *MWT* of a point set in the plane was proposed in [8]; however, the constant is huge and its exact value is unknown.

In contrast, little research has been done on *maximum weight triangulation* (*MAT* for short). From the theoretical viewpoint, the maximum and minimum weight triangulation problems attract equal interest, and one does not seem to be easier than the other. The study of maximum weight triangulations is interesting in theory and may help us to understand the nature of different optimal triangulations.

☆ The first author is supported by NSERC grant OGP0046506 and the National Natural Science Foundation of China under Grant No. 70221001, the second author is supported by NSERC grant OPG0041629.
* Corresponding author.
E-mail addresses: j3qian@cs.uwaterloo.ca (J. Qian), wang@garfield.cs.mun.ca (C.A. Wang).

0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2005.11.001

The first result on the *MAT* problem [9] showed that for an n-sided polygon P inscribed on a circle, $MAT(P)$ can be found in $O(n^2)$ time. Recently, a factor 6 approximation algorithm to find the *MAT* of a set of points in the plane was proposed in [5].

In this paper, we present in Section 3 a non-trivial upper-bound of the weight of $MAT(S)$ for a set S of n points in the plane. Namely, we show that

$$\omega\big(MAT(S)\big) \leqslant \left((2+2\varepsilon)\cdot n + \frac{\pi(1-2\varepsilon)}{8\varepsilon\sqrt{1-\varepsilon^2}} + \frac{\pi}{2} - 5(\varepsilon+1)\right)D_S,$$

where $\omega(MAT(S))$ is the weight of $MAT(S)$, D_S is the diameter of S and ε is any constant such that $0 < \varepsilon \leqslant \frac{1}{2}$. We also show that when $\varepsilon \to 0$, this upper-bound is tight.

Applying a simple 'spoke-scan' approximation algorithm from [5], we obtain a triangulation $T(S)$ with ratio $(\frac{\omega(MAT(S))}{\omega(T(S))} =)$ 4.238, which improves the previous ratio of 6. Furthermore, let P be a special convex n-sided polygon such that the entire polygon P is contained inside the circle with an edge of P as the diameter. We call such a polygon *semi-circled*. In Section 4, we show that the $MAT(P)$ of a semi-circled convex polygon P can be found in $O(n^2)$ time. Recall that a straightforward dynamic programming method will take $O(n^3)$ to find $MAT(P)$ [3,6].

2. Preliminaries

Let S be a set of points in the plane. A *triangulation* of S, denoted by $T(S)$, is a maximal set of non-crossing line segments with their endpoints in S. It follows that the interior of the convex hull of S is partitioned into non-overlapping triangles. The weight of a triangulation $T(S)$, $\omega(T(S))$, is given by the following equality:

$$\omega\big(T(S)\big) = \sum_{\overline{s_is_j}\varepsilon T(S)} \omega(\overline{s_is_j}),$$

where $\omega(\overline{s_is_j})$ is the Euclidean length of line segment $\overline{s_is_j}$.

A *maximum weight triangulation* of S, $MAT(S)$, is defined as a triangulation of S with the greatest weight.

Let P be a convex polygon (whose vertices form a *convex point set*) and $T(P)$ be its triangulation. A *semi-circled convex polygon* P is a special convex polygon such that its longest edge is the diameter and all the edges of P lie inside the semi-circle with this longest edge as its diameter. (Refer to Fig. 1.)

The following two properties are easy to verify for a semi-circled convex polygon $P = (p_1, p_2, \ldots, p_k, \ldots, p_{n-1}, p_n)$.

Property 1. *Let $\overline{p_ip_k}$ for $1 \leqslant i < k \leqslant n$ be a diagonal in P. Then the area bounded by $\overline{p_ip_k}$ and chain $(p_i, \ldots, p_k)$ is a semi-circled convex polygon.*

Property 2. *In P, edge $\overline{p_ip_j}$ for $1 \leqslant i < j < n$ or $1 < i < j \leqslant n$ is shorter than $\overline{p_1p_n}$.*

The 'spoke-scan' approximation algorithm proposed in [5] can be described as follows. Let $D_S = \overline{s_is_j}$ be the diameter of a point set S. The line extending D_S partitions S into S^1 and S^2 (excluding s_i and s_j). Let $\omega(E_i^1)$ and $\omega(E_j^1)$ (respectively, $\omega(E_i^2)$ and $\omega(E_j^2)$) denote the sum of the lengths of edges connecting every points in S^1 to s_i and

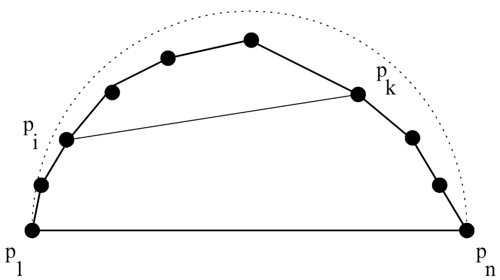

Fig. 1. An illustration of semi-circled convex polygon.

to s_j, respectively. The algorithm first takes the spokes in the larger one of $\omega(E_i^1)$ and $\omega(E_j^1)$ (respectively, the larger one of $\omega(E_i^2)$ and $\omega(E_j^2)$) as the first subset of the triangulation edges. Then, it uses Graham's scan algorithm [4] to complete the triangulation. It is shown that this triangulation has weight at least $(n+1)\frac{D_S}{2}$. We shall use this algorithm in our paper.

3. Tight upper-bound for maximum-weight triangulations

Let D_S denote the diameter of a finite point set S in the plane, let A_S denote the area bounded by the convex hull $CH(S)$, and let L_S denote the perimeter of $CH(S)$. Let the constant ε be in range $0 < \varepsilon \leqslant \frac{1}{2}$.

It is known (see [1,2]) that the value of A_S and L_S is bounded from above by $\frac{\pi}{4}D_S^2$ and πD_S, respectively.

The following lemma demonstrates the relationship between the area of a triangle, $A_\triangle$, the perimeter of the triangle, $\omega(\triangle)$, and the diameter of the triangle, $D_\triangle$.

Lemma 1. *For a given constant ε: $0 < \varepsilon \leqslant \frac{1}{2}$, if perimeter $\omega(\triangle)$ satisfies: $\omega(\triangle) \geqslant (2+2\varepsilon)D_\triangle$, then the area $A_\triangle \geqslant \varepsilon\sqrt{1-\varepsilon^2}\cdot D_\triangle^2$.*

Proof. Let $a, b,$ and c be the lengths of the three edges of a triangle $\triangle$. We shall first prove that if $A_\triangle$ is minimized, then two of $a, b,$ and c equal $D_\triangle$, and the third one equals 2ε.

Note that, when $A_\triangle$ is minimized, the equality

$$a+b+c = (2+2\varepsilon)D_\triangle \tag{1}$$

must hold. This is because otherwise (i.e., $a+b+c > (2+2\varepsilon)D_\triangle$), we can always "shrink" the triangle to reduce its area but without reducing $D_\triangle$ until the equality (1) holds, which contradicts the assumption of the minimality of $A_\triangle$.

Let $l = \frac{a+b+c}{2}$. By Heron's formula, $A_\triangle = \sqrt{l(l-a)(l-b)(l-c)}$.

Since $D_\triangle$ must be equal to one of a, b and c, without loss of generality, we assume $a = D_\triangle$. We claim that if $A_\triangle$ is minimized, then b or c is maximized, that is, one of b and c equals $D_\triangle$. In fact, the following equality holds.

$$(l-b)(l-c) = \frac{(((l-b)+(l-c))^2 - ((l-b)-(l-c))^2)}{4} = \frac{((2l-(b+c))^2-(b-c)^2)}{4}.$$

Then,

$$A_\triangle = \sqrt{l(l-a)\cdot\frac{((2l-(b+c))^2-(b-c)^2)}{4}}. \tag{2}$$

Since l and a are fixed, $(b+c)$ is also fixed. So when $A_\triangle$ is minimized, $(b-c)^2$ in (2) must be maximized, that is, b or c must equal $D_\triangle$.

Therefore, when $A_\triangle$ is minimized, two of $a, b,$ and c equal $D_\triangle$, and by $a+b+c = (2+2\varepsilon)D_\triangle$, the third one equals $2\varepsilon D_\triangle$.

By (1) and definition of l, $l = (1+\varepsilon)\cdot D_\triangle$. Without loss of generality, let $b = D_\triangle$, we have

$$A_\triangle = \sqrt{(1+\varepsilon)D_\triangle\cdot\varepsilon D_\triangle\cdot\varepsilon D_\triangle\cdot(1-\varepsilon)D_\triangle} = \varepsilon\sqrt{1-\varepsilon^2}D_\triangle^2.$$

We now have $A_\triangle \geqslant \varepsilon\sqrt{1-\varepsilon^2}D_\triangle^2$ for $0 < \varepsilon \leqslant \frac{1}{2}$. □

In a triangulation of S, $T(S)$, we call a triangle t 'large' if the perimeter of t, $\omega(t)$ is at least $(2+2\varepsilon)\cdot D_S$; and t is 'small', otherwise. By Lemma 1 and by the fact that the maximum area of the convex hull of a point set with diameter D_S is bounded above by $\frac{\pi}{4}D_S^2$ [1,2], and note that $D_\triangle$ is at most D_S, we have an upper-bound for the number of large triangles in $T(S)$:

Lemma 2. *$T(S)$ may contain at most $\frac{\pi}{4\varepsilon\sqrt{1-\varepsilon^2}}$ large triangles.*

Now we can derive the upper-bound on the length, $\omega(T)$, of $T(S)$:

Theorem 1. $\omega(T) \leqslant ((2+2\varepsilon)\cdot n + \frac{\pi(1-2\varepsilon)}{8\varepsilon\sqrt{1-\varepsilon^2}} + \frac{\pi}{2} - 5(1+\varepsilon))D_S$.

Proof. Let m denote the number of large triangles. It is well known that $T(S)$ contains at most $2n-5$ triangles. Thus, $T(S)$ has at most $2n-5-m$ small triangles. It is clear that the perimeter of a large triangle is at most $3D_S$, and the perimeter of a small one is less than $(2+2\varepsilon)\cdot D_S$ by definition. If we add up the perimeters of all the triangles in $T(S)$ as well as the perimeter of $CH(S)$, L_S, we obtain $2\omega(T)$. Thus,

$$\begin{aligned}
\omega(T) &\leqslant \frac{1}{2}\big((2+2\varepsilon)(2n-5-m)D_S + 3mD_S + L_S\big) \\
&\leqslant \frac{1}{2}\left((2+2\varepsilon)\left(2n-5-\frac{\pi}{4\varepsilon\sqrt{1-\varepsilon^2}}\right) + 3\frac{\pi}{4\varepsilon\sqrt{1-\varepsilon^2}} + \pi\right)D_S \\
&\quad \left(\text{note that } m \leqslant \frac{\pi}{4\varepsilon\sqrt{1-\varepsilon^2}},\ 0<\varepsilon\leqslant\frac{1}{2},\ \text{and } L_S \leqslant \pi D_S\right) \\
&= (2+2\varepsilon)D_S\cdot n + \left(\frac{\pi(1-2\varepsilon)}{8\varepsilon\sqrt{1-\varepsilon^2}} + \frac{\pi}{2} - 5(1+\varepsilon)\right)D_S. \qquad \square
\end{aligned}$$

Remark 1. Note that the above upper-bound depends on parameter ε. When $\varepsilon = \frac{1}{2}$, our bound is very close to an obvious upper-bound, $3n-6$ for $D_S = 1$. On the other hand, when ε is sufficiently small, the upper-bound in Theorem 1 is the best possible one with the measurement of D_S. This is because there exists a point set whose *MAT* has weight $(2 - \frac{2\sqrt{3}\delta}{D_S})D_S\cdot n - 3(D_S - (\sqrt{3}+1)\delta)$ for any constant $\delta > 0$. When δ is sufficiently small, the weight approaches the upper-bound.

For more details about the claim above, let us consider the following point set.

We start by placing three points on the vertices of an equilateral triangle, say s_1, s_2, and s_3. Let o be the center of the circle inscribing at $\triangle s_1s_2s_3$. Now, we place $\frac{n}{3}-1$ points evenly on the line segment $\overline{os_1}$ such that the distance from s_1 to the farthest point is δ. Similarly, we place $\frac{n}{3}-1$ points on the line segment $\overline{os_2}$ and $\frac{n}{3}-1$ points on the line segment $\overline{os_3}$ in the same manner. (Refer to part (a) of Fig. 2.) To construct a triangulation, we connect all points in the neighbouring groups by edges. Thus, we obtain a triangulation T' with $n-3$ short edges and $2n-3$ long edges, where the sum of the lengths of the shorter edges is no less than 3δ and the sum of the lengths of the long edges is greater

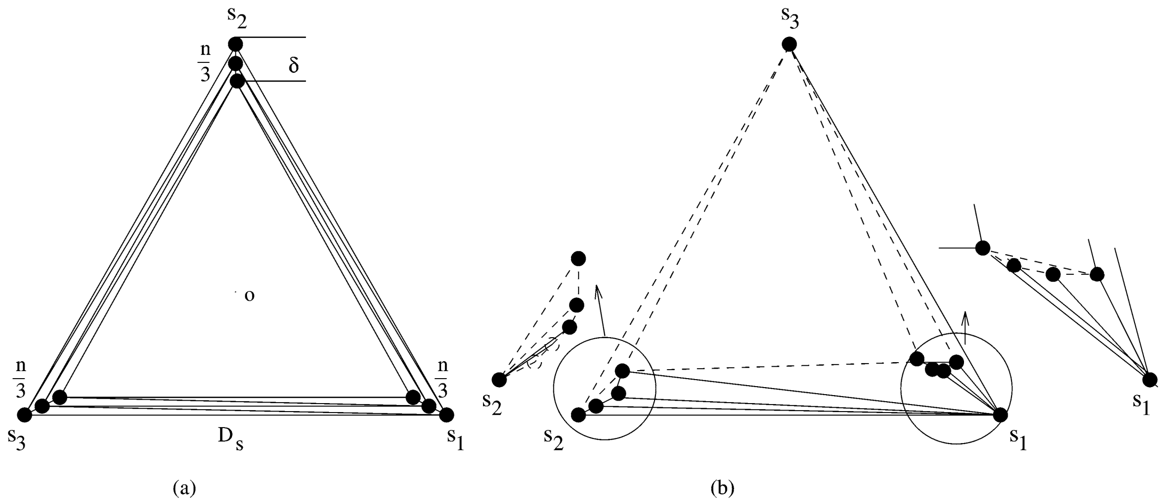

Fig. 2. An illustration of the tight upper-bound (a), and a bad case for 'spoke-scan' algorithm (b).

than $(2n-3)\cdot(D_S-\sqrt{3}\delta)$. Then, the total length of the triangulation, $\omega(T')$, is at least $(2n-3)(D_S-\sqrt{3}\delta)+3\delta$. Then,

$$\omega(T') \geqslant 2nD_S - 2n\sqrt{3}\delta - 3D_S + -3\sqrt{3}\delta + 3\delta = \left(2-\frac{2\sqrt{3}\delta}{D_S}\right)D_S\cdot n - 3\bigl(D_S-(\sqrt{3}+1)\delta\bigr).$$

Recall that the approximation spoke-scan algorithm produces a triangulation with length at least $(n+1)\cdot\frac{D_S}{2}$.

If we apply this spoke-scan algorithm to produce a triangulation of any point set, then by our upper-bound, we obtain a performance ratio as stated in the following theorem.

Theorem 2. $\frac{\omega(T(S))}{\omega(T'(S))} \to_{n\to\infty} 4+4\varepsilon$ *for* $0<\varepsilon\leqslant\frac{1}{2}$, *where* $T'(S)$ *is produced by 'spoke-scan' approximation algorithm and* $\omega(T(S))$ *is the upper-bound in Theorem* 1.

Proof.

$$\frac{\omega(T(S))}{\omega(T'(S))} = \frac{2n(1+\varepsilon)D_S + (\frac{\pi(1-2\varepsilon)}{8\varepsilon\sqrt{1-\varepsilon^2}} + \frac{\pi}{2} - 5(1+\varepsilon))\cdot D_S}{(n+1)\cdot\frac{D_S}{2}} = (4+4\varepsilon) + \frac{b(\varepsilon)}{n+1},$$

where

$$b(\varepsilon) = \frac{\pi(1-2\varepsilon)}{4\varepsilon\sqrt{1-\varepsilon^2}} + \pi - 14(1+\varepsilon).$$

When n is sufficiently large, we obtain the desired ratio. □

Remark 2. When the value of ε is properly chosen, the $4+4\varepsilon$ ratio will apply to any n. To see this, let $\frac{\omega(T(S))}{\omega(T'(S))}\leqslant 4+4\varepsilon$, that is,

$$\frac{2n(1+\varepsilon)D_S + (\frac{\pi(1-2\varepsilon)}{8\varepsilon\sqrt{1-\varepsilon^2}} + \frac{\pi}{2} - 5(1+\varepsilon))\cdot D_S}{(n+1)\cdot\frac{D_S}{2}} \leqslant 4+4\varepsilon$$

for any $n>0$. We have $2(\frac{\pi(1-2\varepsilon)}{8\varepsilon\sqrt{1-\varepsilon^2}} + \frac{\pi}{2} - 5(1+\varepsilon)) \leqslant 4+4\varepsilon$ after simplification. Solving this inequality for ε, we have $\varepsilon \geqslant .05932390326$.

Therefore, when we apply the 'spoke-scan' algorithm to any point set with $n\geqslant 3$, we have an approximation ratio of 4.238.

4. Finding an MAT of a semi-circled convex polygon

We first prove the following result:

Lemma 3. *Let* $P=(p_1,p_2,\ldots,p_k,\ldots,p_{n-1},p_n)$ *be a semi-circled convex polygon. Then, one of the edges,* $\overline{p_1p_{n-1}}$ *or* $\overline{p_np_2}$, *must belong to its maximum weight triangulation* $MAT(P)$.

Proof. Suppose for contradiction that neither of the two extreme edges, $\overline{p_1p_{n-1}}$ and $\overline{p_np_2}$, belong to $MAT(P)$. Then, there must exist a vertex p_k for $2<k<n-1$ such that both $\overline{p_1p_k}$ and $\overline{p_np_k}$ belong to $MAT(P)$. Without loss of generality, let p_k lie on one side of the perpendicular bisector of edge $\overline{p_1p_n}$, say the left-hand side (if p_k is on the bisector, the following argument is still applicable). The two edges, $\overline{p_1p_k}$ and $\overline{p_np_k}$, partition P into three areas, denoted by R, L, and C. Both L and R are semi-circled convex polygons by Property 1. (Refer to part (a) of Fig. 3.)

Let the edges of $MAT(P)$ lying inside area L be $(e_1,e_2,\ldots,e_{k-3})$ and let $E_L=(e_1,e_2,\ldots,e_{k-3},\overline{p_1p_k})$. Let E_L^* denote the edges: $(\overline{p_np_2},\overline{p_np_3},\ldots,\overline{p_np_{k-1}})$. Then, there is a perfect matching between E_L and E_L^*. This is because E_L and E_L^* respectively triangulate the same area $L\cup C$, hence the number of internal edges of the two triangulations must be equal. Let us consider a pair in the matching $(e_i,\overline{p_np_j})$ for $1<i,j<k$. It is not hard to see that $\omega(e_i)<\omega(\overline{p_np_j})$ because any edge in E_L is shorter than $\overline{p_1p_k}$ by Property 2, any edge in E_L^* is longer than $\overline{p_np_k}$, and $\overline{p_np_k}$ is longer than $\overline{p_1p_k}$ (due to p_k lying on the left-hand side of the perpendicular bisector of $\overline{p_1p_n}$).

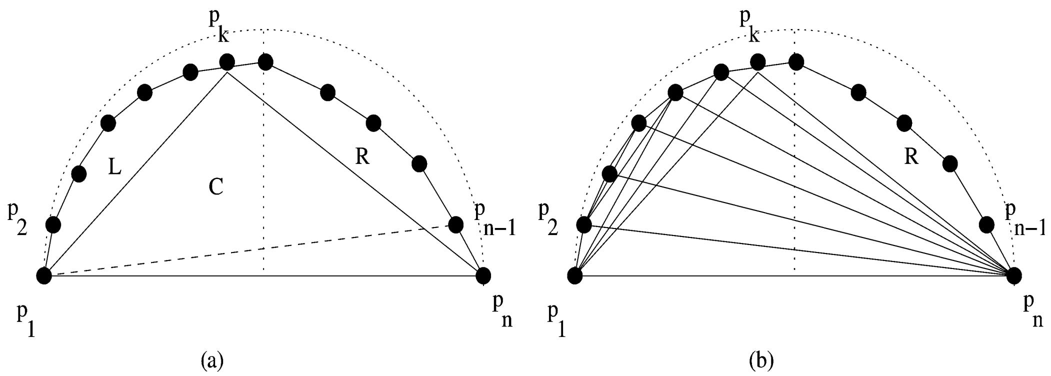

Fig. 3. Illustration of the proof of Lemma 3.

Therefore, $\omega(E_L)$ is less than $\omega(E_L^*)$. Now, we shall construct a new triangulation, say $T(P)$, which consists of all the edges in $MAT(P)$ except replacing the edges of E_L by E_L^*. We have $\omega(T(P)) > \omega(MAT(P))$, which contradicts the $MAT(P)$ assumption. Then, such a p_k cannot exist and one of $\overline{p_1 p_{n-1}}$ and $\overline{p_n p_2}$ must belong to $MAT(P)$. □

By Lemma 3 and by Property 1, we have a recurrence for the weight of a $MAT(P)$. Let $\omega(i, j)$ denote the weight of the $MAT(P_{i,j})$ of a semi-circled convex polygon $P_{i,j} = (p_i, p_{i+1}, \ldots, p_j)$. Let $\omega(\overline{p_i p_j})$ denote the length of edge $\overline{p_i p_j}$.

$$\omega(i, j) = \begin{cases} \omega(\overline{p_i p_{i+1}}), & j = (i+1), \\ \max\{\omega(i, j-1) + \omega(\overline{p_{j-1} p_j}), \omega(i+1, j) + \omega(\overline{p_i p_{i+1}})\} + \omega(\overline{p_i p_j}), & \text{otherwise.} \end{cases}$$

It is a straightforward matter to design a $O(n^2)$ dynamic programming algorithm for finding the $MAT(P)$ given this recurrence. Therefore, we have

Theorem 3. *The maximum weight triangulation of a semi-circled convex n-gon P, $MAT(P)$, can be found in* $O(n^2)$ *time.*

5. Conclusion

In this paper, we derived a non-trivial tight upper-bound of the maximum weight triangulation of a point set in the plane. We also used the 'spoke-scan' algorithm to obtain a factor 4.238 triangulation for any point set in the plane, which improved the previous result: factor 6. We finally proposed an $O(n^2)$ dynamic programming algorithm for constructing the $MAT(P)$ of a semi-circled convex n-sided polygon.

It is interesting to see the range of the performance ratio of the spoke-scan algorithm. There is a good case in which the algorithm produces an optimal solution of $(\frac{D}{2} + \delta)(n+1))$ for the following point set.

Given any constant $\delta > 0$, we place n points on the unit interval. Two of them were located at the ends of the interval and the rest are in interval $(\frac{1}{2} - \delta, \frac{1}{2} + \delta)$. Then raise the points near the center slightly so that the n points form a convex polygon, and every point near the center is at most $\frac{1}{2} + \delta$ to any of the end points. (Refer to Fig. 4.) It is not difficult to see that any triangulation of these n points has length at most $(\frac{D}{2} + \delta)(n+1)$.

There is a bad case in which the spoke-scan algorithm produces a solution with ratio 4 (refer to part (b) of Fig. 2). We place three points on the vertices of an equilateral triangle, say s_1, s_2, and s_3 respectively, then place $\frac{n-3}{2}$ points near s_1, such that the distance from s_1 to the farthest point is δ. Similarly, we place $\frac{n-3}{2}$ points near s_2 in the same

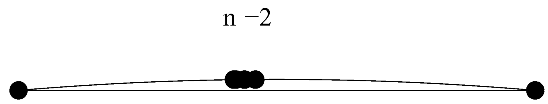

Fig. 4. An illustration of the good case.

manner. We slightly perturb these points such that the sum of the lengths of spokes from s_1 is greater than that from s_2. Thus, these n points were divided into three groups: s_1 and $\frac{n-3}{2}$ points, s_2 and $\frac{n-3}{2}$ points, and s_3. We then arrange each group of $\frac{n-1}{2}$ points so that, during the scan stage, there is only 5 long edges created (indicated by dashed lines). It is not difficult to see that the optimal solution is at least $(2n-3)(D_S-\sqrt{3}\delta)$ and the spoke-scan algorithm produces a solution with weight at most $\frac{n+11}{2}D_S+3n\delta$ (the spoke from s_1 consists of $\frac{n+1}{2}$ long edges). The ratio approaches a value of 4 for sufficiently small δ and large n.

It is still an open problem whether one can design an $o(n^3)$ algorithm for finding the $MAT(P)$ for a general convex n-sided polygon P.

References

[1] P.R. Scott, P.W. Awyong, Inequalities for convex sets, J. Inequal. Pure Appl. Math. 1 (1) (2000) 6, Article.

[2] I.M. Yaglom, V.G. Boltyanskli, Convex Figures, Holt, Rinehart and Winston, New York, 1961, translated by P.J. Kelly and L.F. Walton.

[3] P. Gilbert, New results on planar triangulations, Tech. Rep. ACT-15, Coord. Sci. Lab., University of Illinois at Urbana, 1979.

[4] R.L. Graham, An efficient algorithm for determining the convex hull of a finite planar set, Inform. Process. Lett. 1 (4) (1972) 132–133.

[5] S. Hu, A constant-factor approximation for maximum weight triangulation, in: Proceedings of 15th Canadian Conference of Computational Geometry, Halifax, NS, Canada, pp. 150–154.

[6] G. Klincsek, Minimal triangulations of polygonal domains, Annual Discrete Math. 9 (1980) 121–123.

[7] C. Levcopoulos, A. Lingas, On approximation behavior of the greedy triangulation for convex polygon, Algorithmica 2 (1987) 175–193.

[8] C. Levcopoulos, D. Krznaric, Quasi-greedy triangulations approximating the minimum weight triangulation, in: Proceedings of the 7th ACM-SIAM Symposium on Discrete Algorithms (SODA), 1996, pp. 392–401.

[9] C. Wang, F. Chin, B. Yang, Maximum weight triangulation and graph drawing, Inform. Process. Lett. 70 (1999) 17–22.